## Exercise 2.4.7

Use linear stability analysis to classify the fixed points of the following systems. If linear stability analysis fails because $f^{\prime}\left(x^{*}\right)=0$, use a graphical argument to decide the stability.

$$
\dot{x}=a x-x^{3} \text {, where } a \text { can be positive, negative, or zero. Discuss all three cases. }
$$

## Solution

Case I: $a>0$
The fixed points occur where $\dot{x}=0$.

$$
\begin{gathered}
a x^{*}-x^{* 3}=0 \\
x^{*}\left(a-x^{* 2}\right)=0 \\
x^{*}\left(\sqrt{a}+x^{*}\right)\left(\sqrt{a}-x^{*}\right)=0 \\
x^{*}=0 \quad \text { or } \quad x^{*}=-\sqrt{a} \quad \text { or } \quad x^{*}=\sqrt{a}
\end{gathered}
$$

Apply linear stability analysis to determine whether each of these points is stable or unstable.

$$
f(x)=a x-x^{3}
$$

Differentiate $f(x)$.

$$
f^{\prime}(x)=a-3 x^{2}
$$

As a result,

$$
\begin{aligned}
f^{\prime}(-\sqrt{a}) & =-2 a<0 & \Rightarrow & x^{*}=-\sqrt{a} \text { is a stable fixed point. } \\
f^{\prime}(0) & =a>0 & \Rightarrow & x^{*}=0 \text { is an unstable fixed point. } \\
f^{\prime}(\sqrt{a}) & =-2 a<0 & \Rightarrow & x^{*}=\sqrt{a} \text { is a stable fixed point. }
\end{aligned}
$$

The graph of $\dot{x}$ versus $x$ below confirms these results.


Case II: $a<0$
The fixed points occur where $\dot{x}=0$.

$$
\begin{gathered}
a x^{*}-x^{* 3}=0 \\
x^{*}\left(a-x^{* 2}\right)=0 \\
x^{*}=0
\end{gathered}
$$

Apply linear stability analysis to determine whether this point is stable or unstable.

$$
f(x)=a x-x^{3}
$$

Differentiate $f(x)$.

$$
f^{\prime}(x)=a-3 x^{2}
$$

As a result,

$$
f^{\prime}(0)=a<0 \quad \Rightarrow \quad x^{*}=0 \text { is a stable fixed point. }
$$

The graph of $\dot{x}$ versus $x$ below confirms it.


Case III: $a=0$
The fixed points occur where $\dot{x}=0$.

$$
\begin{gathered}
-x^{* 3}=0 \\
x^{*}=0
\end{gathered}
$$

Apply linear stability analysis to determine whether this point is stable or unstable.

$$
f(x)=-x^{3}
$$

Differentiate $f(x)$.

$$
f^{\prime}(x)=-3 x^{2}
$$

As a result,

$$
f^{\prime}(0)=0 \quad \Rightarrow \quad \text { No conclusion can be made about the stability of } x^{*}=0
$$

The graph of $\dot{x}$ versus $x$ below shows that $x^{*}=0$ is a stable fixed point.


