Use linear stability analysis to classify the fixed points of the following systems. If linear stability analysis fails because $f'(x^*) = 0$, use a graphical argument to decide the stability.

 $\dot{x} = ax - x^3$, where a can be positive, negative, or zero. Discuss all three cases.

Solution

Case I: a > 0

The fixed points occur where $\dot{x} = 0$.

$$ax^{*} - x^{*3} = 0$$
$$x^{*}(a - x^{*2}) = 0$$
$$x^{*}(\sqrt{a} + x^{*})(\sqrt{a} - x^{*}) = 0$$
$$x^{*} = 0 \quad \text{or} \quad x^{*} = -\sqrt{a} \quad \text{or} \quad x^{*} = \sqrt{a}$$

Apply linear stability analysis to determine whether each of these points is stable or unstable.

$$f(x) = ax - x^3$$

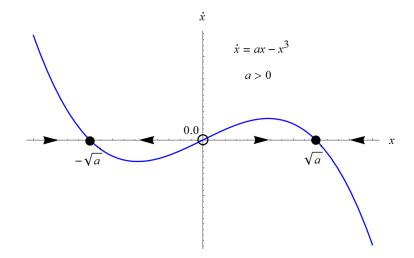
Differentiate f(x).

$$f'(x) = a - 3x^2$$

As a result,

$$f'(-\sqrt{a}) = -2a < 0 \qquad \Rightarrow \qquad x^* = -\sqrt{a}$$
 is a stable fixed point.
 $f'(0) = a > 0 \qquad \Rightarrow \qquad x^* = 0$ is an unstable fixed point.
 $f'(\sqrt{a}) = -2a < 0 \qquad \Rightarrow \qquad x^* = \sqrt{a}$ is a stable fixed point.

The graph of \dot{x} versus x below confirms these results.



Case II: a < 0

The fixed points occur where $\dot{x} = 0$.

$$ax^* - x^{*3} = 0$$
$$x^*(a - x^{*2}) = 0$$
$$x^* = 0$$

Apply linear stability analysis to determine whether this point is stable or unstable.

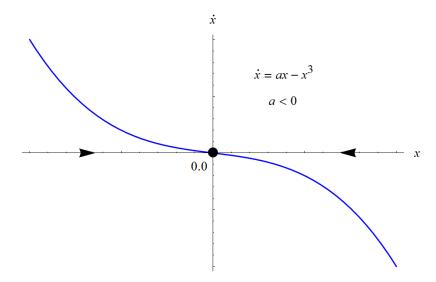
$$f(x) = ax - x^3$$
$$f'(x) = a - 3x^2$$

Differentiate f(x).

As a result,

 $f'(0) = a < 0 \qquad \Rightarrow \qquad x^* = 0$ is a stable fixed point.

The graph of \dot{x} versus x below confirms it.



Case III: a = 0

The fixed points occur where $\dot{x} = 0$.

$$-x^{*3} = 0$$
$$x^* = 0$$

Apply linear stability analysis to determine whether this point is stable or unstable.

$$f(x) = -x^3$$
$$f'(x) = -3x^2$$

As a result,

Differentiate f(x).

 $f'(0) = 0 \implies$ No conclusion can be made about the stability of $x^* = 0$.

The graph of \dot{x} versus x below shows that $x^* = 0$ is a stable fixed point.

